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A Generlisation of Classical Probability

Abstract

It is remarkable and uncontroversial that quantum mechanics needs neatly a generalisation of classical probability in which the role played by Boolean σ-algebra of events is taken over the quantum logic of projection operators on a Hilbert space. The probability on a quantum logic is more complex than its classical concept on a Boolean σ-algebra. In this paper our purpose is to introduce a new concept of probability on a lattice L which is a generalisation of the classical theory.

Keywords: Orthomodular lattice, pseudocomplemented lattice, valuation, orthologic, continuous geometry.

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Introduction

In 1936 Birkhoff and von Neumann discovered that the experimental propositions of quantum system represented by the set of all projections in a seperable, infinite dimensional complex Hilbert space (or equivatently by that of its all closed subspaces) was not a Boolean σ-algebra but a highly non-distributive orthocomplemented lattice [7]. It should be noted that the collection of all projections on a Hilbert space form a complete orthomodular atomic order symmetric lattice, which is of course not modular. But they did not abandon the idea that the lattice of quantum mechanical propositions was modular [5]. On the other hand if a logic is modular, and separable, then it follows from a general result of Kaplanasky [19] that it is a continuous geometry in the sense of von Neumann. Whether the logic of any atomic system may be assumed to be an orthocomplementad continuous geometry is still an open question. Such a logic was investigated by several practitioners of quantum mechanics [1, 2, 8-17, 20-22, 24-33]. Randall and Foulis [26] introduced a new type or logic called an orthologic, one in which the execution of more than one physical experiments is permissible. This logic was then investigated by Barbara [3, 4]. Accordi Luigi [20] provides a historical account of development of a coherent quantum probabilistic approach to the foundation of quantum mechanics. According to Plotnitsky quantum mechanics is a probabilistic theory of individual quantum events rather than a statistical theory of ensemble [25]. A. Doring and C. Isham [11] considered intuitionistic logic (i.e. dual Broweian logic) to study a non-classical physical system. Almost all logics associated with quantum mechanics are orthologic in which the notion of compatibility can be introduced in the same way as it was introduced by Mackey [21, p. 70] and the notion of observable in the same way as by Varadarajan [29, p. 108].

Our endeavour here is to study the theory of probability on a pseudocomplemented lattice which is a basic ingredient of intuitionistic logic. The terminolosics used in this paper are available in [5, 23, 18].

1. Probability on a Lattice

We introduce here the notion of probability on an arbitrary bounded lattice which may be useful to study a non-classical physical system.

Definition 1.1 : A probability lattice denoted by p-lattice is a lattice L with a real valued function p on L satisfying the following properties :

(i) p is strictly positive, i.e., $p(x) > 0$, $\forall x \in L$ and $p(x) = 0$ iff $x = 0$ where 0 is the zero element of L.

(ii) p is normed, i.e., $e \in L \Rightarrow p(e) = 1$, where e is the unit element of L;

(iii) p is additive,

i.e. $p(a \vee b) = p(a) + p(b)$, where a and b are disjoint elements of L.

Theorem 1.1. (i) p is monotonic

(ii) The probability on a pseudocomplemented lattice is a valuation.

(iii) $p(x^*) = 1 - p(x)$ where x^* is the pseudocomplement of x in L.

(iv) If $\bar{a}$ is the local closure of $a \in L$,

i.e. $\bar{a} = \vee\{x^{**} : x \leq a\}$. Then $p(\bar{a}) = 1 - p(a^*)$.

Proof. (i) If $z = y \wedge x^*$, then $y = x \vee z$ where x and z are disjoint.

We have $p(y) = p(x) + p(z)$

i.e. $p(x) \le p(y)$

If $x \ne y$, then $z \ne 0$ and $p(z) > 0$

i.e. $p(x) < p(y)$

(ii) Here we shall prove that for every pair $x, y \in L$,

$p(x) + p(y) = p(x \vee y) + p(x \wedge y)$

We have $x \vee y = (x \wedge y^*) \vee (y \wedge x^*) \wedge (x \vee y)$.

The terms of the join under the parentheses on the right are pairwise disjoint.

$p(x \vee y) = p(x \wedge y^*) + p(y \wedge x^*) + p(x \vee y)$

Now, $x = (x \wedge y^*) \vee (x \wedge y)$

$\therefore\ p(x) = p(x \wedge y^*) + p(x \wedge y)$

$\Rightarrow p(x \wedge y^*) = p(x) - p(x \wedge y)$

$\therefore\ p(x \vee y) = p(x) - p(x \wedge y) + p(y) - p(x \wedge y) + p(x \wedge y)$

i.e. $p(x) + p(y) = p(x \vee y) + p(x \wedge y)$

(iii) $x \wedge x^* = 0$

$\Rightarrow p(x \vee x^*) = p(x) + p(x^*)$

$\Rightarrow p(e) = p(x) + p(x^*)$

$\Rightarrow p(x^*) = 1 - p(x)$

(iv) Remember that an element k of a lattice L is compact if for any set $X \subseteq L$ such that $k \le \vee X$, there exists a finite $Y \subseteq X$ such that $k \le \vee Y$. An element $a \in L$ is locally closed if for each compact element $k \le a$ it holds $k^{**} \le a$. Further we put $\bar{a} = \vee \{k^{**} : k \le a\}$ and call $\bar{a}$ the local closure of a. Obviously every closed element is locally closed. If $a \in L$. Then $\bar{a}$ is the least locally closed element $\ge a$ and the map $x \to \overline{x}$ is a closure operation on L. To prove this result we see that if $a \le b$ and b is a locally closed element. Then $\bar{a} \le b$. Let us show that $\bar{a}$ is locally closed. Let $l \le \bar{a}$ be compact. Then there is a $k \le a$ such that $l \le k^{**}$ and consequently $l^{**} \le k^{**} \le \bar{a}$. The last assertion is trivially true. Hence a is locally closed iff $a = \bar{a}$. The rest of the proof is a direct consequence of the previous result.

Definition 1.2. Let (L, p) be a probability lattice and L_0, a sublattice of L, then the restriction of p to L_0 is a probability on L_0. The probability lattice (L_0, p) is then called a probability sublattice of (L, p).

If L_0 is a non-empty subset of L, then there exists a smallest sublattice of L containing L_0, the probability lattice is then called p-sublattice generated by L_0 in (L, p).

Definition 1.3. A lattice L is isomorphic to a set lattice of all principal ideals generated by each element of L which with a set probability p is called a lattice probability space.

Definition 1.4. Let (L, p) be a p-lattice. The probability p is said to be countably additive or σ-additive on L iff for every countable sequence a_v, $v = 1, 2, \ldots$ of pairwise disjoint elements in L, we have

$$\bigvee_{\nu=1}^{\infty} a_\nu = a \text{, exists and}$$

$$\sum_{\nu=1}^{\infty} p(a_\nu) = p(a)$$

A p-lattice is said to be p-σ lattice iff the lattice L is a σ-lattice and probability p is σ-additive on L.

2. Homometrization and seperability

Definition 2.1. A probability lattice (L_1, p_1) is said to homometric to (L_2, p_2) iff there exists a mapping

$$f : (L_1, p_1) \to (L_2, p_2)$$

such that f is a lattice homomorphism and

$$p_1(x) = p_2(f(x))$$

If the mapping f is a lattice isomorphism, then one of the p-lattice is said to be isometric with others.

Definition 2.2. Let (L, p) be a probability lattice and L_0 a subset of L, then we say that L_0 is p-dense in L, iff for every $x \in L$ and for every positive real number $\varepsilon > 0$, there exists an element $a = a(x, \varepsilon) \in L_0$, such that $p(x \vee a) < \varepsilon$. A p-lattice (L, p) is called p-separable iff there exists a countable class C of elements of L, which is p-dense in L.

Every p-sublattice of p-separable p-lattice is also p-separable.

Theorem 2.1. The probability interval lattice (L, m) is m-seperable.

Proof. Let L_0 be a sublattice of L generated by the class of all intervals I_α for every α. Then L_0 is a countable set and it is m-separable.

Theorem 2.2. Let (L, p) be pseudocomplemented probability lattice. Let ρ be a real valued function defined on $L \times L$ as follows:

$\rho(a, b) = p(a \vee b)$

and $\rho(a, b) = 0$ iff $a = b$.

Then the following conditions hold for all $a, b, c \in L$

(i) $\rho(a, b) \ge 0$ and $\rho(a, b) = 0$ iff $a = b$

(ii) $\rho(a, b) = \rho(b, a)$

(iii) $\rho(a, b) \le \rho(a, c) + \rho(c, b)$

Proof. Here (i) and (ii) are trivially true.

We shall prove (iii)

We have $\rho(a, b) = p(a \vee b) \le p(a) + p(b)$

Also, $p(a) + p(b) \le \rho(a \vee c) + \rho(c \vee b)$

i.e. $\rho(a, b) \le \rho(a, c) + \rho(c, b)$

Hence (iii) is true.

Hence the lattice L can be considered as a metric topological space and the concept of metric convergence or equivalently p-convergence can be introduced in the usual way, namely, a sequence $a_\nu \in L$, $\nu = 1, 2, \ldots$ is said to p-convergent to an element a if and only if

$$\lim_{\nu\to\infty} a_\nu = a$$

A p-convergent sequence $a_\nu \in L$, $\nu = 1, 2, \ldots$ satisfies the p-cauchy condition i.e. for every $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that

$$p(a_v \vee a_u) < \varepsilon$$

for every $u, v \ge N(\varepsilon)$, $u = v + p$, and p is a positive integer ≥ 1.

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